

Geometrical Features of $(4 + d)$ Gravity

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Abstract

We obtain the vacuum spherical symmetric solutions for the gravitational sector of a $(4 + d)$ -dimensional Kaluza-Klein theory. In the various regions of parameter space, the solutions can describe either naked singularities or black-holes or wormholes. We also derive, by performing a conformal rescaling, the corresponding picture in the four-dimensional space-time.

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1 Introduction

In recent years, after the great deal of activity since the seventies, the Kaluza-Klein [1] idea of extra dimensions has been the subject of revived interest, and the role of compactified dimensions in physics has appeared in a modern perspective [2].

For instance supersymmetric theories of gravity was particularly successful in $D = 11$: in fact eleven is the maximum number of dimensions consistent with a single graviton [3] and eleven is also the minimum number of dimensions required to unify all the forces in the standard model [4].

An appealing aim of Kaluza-Klein theories in $(4 + d)$ dimensions, where d is the number of compactified dimensions, is to construct a pure geometrical description of physics as a result of minimal extensions to Einstein's theory of general relativity. It is well known that there are many difficulties in realizing this goal, nevertheless it seems important to explore the properties of these higher dimensional theories and to derive consequences susceptible to experimental test. For a review of higher dimensional unified theories from the point of view of general relativity, we refer the reader to the report of Overduin and Wesson [5].

In this paper we discuss the gravitational sector of the theory. In sect.2 we obtain the spherically symmetric vacuum solutions in a $(4 + d)$ -dimensional space-time. Sect.3 is devoted to the classification of these solutions which, in the various regions of parameter space, can describe either naked singularities, or black-holes or wormholes. In sect.4, by a conformal rescaling, we recover the effective four-dimensional picture and show the equivalence of $(4 + d)$ -dimensional vacuum general relativity and four-dimensional general relativity plus a massless scalar field.

2 The $(4 + d)$ -dimensional metric tensor

We consider a $(4 + d)$ -dimensional vacuum space-time $M_4 \otimes S_d$, where M_4 is the ordinary four-dimensional space-time and S_d is a compact internal manifold with d dimensions.

By introducing coordinates (x^μ, y^i) ($\mu = 1, 2, 3, 4; i = 5, \dots, 4+d$) where the y^i have circular topology $0 \leq y^i \leq 2\pi\rho^i$, and supposing that the functions depend on the coordinates x^μ only, we can write the metric in the form

$$ds_{4+d}^2 = \sum_{\mu, \nu} g_{\mu, \nu}(x) dx^\mu dx^\nu + \sum_i C_i^2(x) dy^i dy^i \quad (1)$$

Because we are not interested here in gauge fields contributions, the extra off-diagonal terms have been dropped.

The metric tensor will be considered static and three-dimensional spherically symmetric so, using isotropic coordinates, the correspondent line element can be written as

$$ds_{4+d}^2 = -A^2(r) dt^2 + B^2(r) (dr^2 + r^2 d\Omega^2) + \sum_i C_i^2(r) dy^i dy^i \quad (2)$$

The vacuum Einstein equations reduce to the following system of coupled ordinary differential equations:

$$\frac{A''}{A'} + \frac{B'}{B} + \frac{2}{r} + \sum_i \frac{C'_i}{C_i} = 0 \quad (3a)$$

$$\frac{A'}{A} + \frac{B'}{B} + \frac{2}{r} + \frac{C''_j}{C'_j} + \sum_i \frac{C'_i}{C_i} - \frac{C'_j}{C_j} = 0 \quad (j = 5, \dots, 4+d) \quad (3b)$$

$$\frac{B''}{B} + \frac{B'}{B} \left(\frac{A'}{A} + \sum_i \frac{C'_i}{C_i} \right) + \frac{1}{r} \left(\frac{A'}{A} + 3 \frac{B'}{B} + \sum_i \frac{C'_i}{C_i} \right) = 0 \quad (3c)$$

$$\frac{A''}{A} + \frac{B''}{B} + \sum_i \frac{C''_i}{C_i} + \left(2 \frac{B'}{B} + \frac{1}{r} \right) \left(\frac{A'}{A} + \frac{B'}{B} + \sum_i \frac{C'_i}{C_i} \right) = 0 \quad (3d)$$

The solutions of system (3) are

$$A(r) = \left| \frac{1 - \frac{\eta}{2r}}{1 + \frac{\eta}{2r}} \right|^{\frac{m}{\eta}} \quad (4)$$

$$B(r) = \left(1 + \frac{\eta}{2r}\right)^2 \left| \frac{1 - \frac{\eta}{2r}}{1 + \frac{\eta}{2r}} \right|^{1 - \frac{m + \sum_i \sigma_i}{\eta}} \quad (5)$$

$$C_i(r) = \left| \frac{1 - \frac{\eta}{2r}}{1 + \frac{\eta}{2r}} \right|^{\frac{\sigma_i}{\eta}} \quad (6)$$

where the parameters m, σ_i and η obey the constraint

$$\eta^2 = \frac{1}{2} \left[m^2 + \sum_i \sigma_i^2 + \left(m + \sum_i \sigma_i \right)^2 \right] \quad (7)$$

Thus the line element (2) becomes

$$\begin{aligned} ds_{4+d}^2 = & - \left| \frac{1 - \frac{\eta}{2r}}{1 + \frac{\eta}{2r}} \right|^{2\frac{m}{\eta}} dt^2 + \left(1 + \frac{\eta}{2r}\right)^4 \left| \frac{1 - \frac{\eta}{2r}}{1 + \frac{\eta}{2r}} \right|^{2\left(1 - \frac{m + \sum_i \sigma_i}{\eta}\right)} (dr^2 + r^2 d\Omega^2) \\ & + \sum_i \left| \frac{1 - \frac{\eta}{2r}}{1 + \frac{\eta}{2r}} \right|^{2\frac{\sigma_i}{\eta}} dy^i dy^i \end{aligned} \quad (8)$$

Analizing the behavior of the above line element at spatial infinity and being aware that in $(4 + d)$ dimensions the gravitational coupling constant G_{4+d} is related to the Newton constant G by

$$G = \frac{G_{4+d}}{(2\pi)^d \prod_i \rho_i} \quad (9)$$

the parameter m can be identified with the mass and so must be nonnegative, while each of the quantities σ_i can be of either sign.

We remind that static and spherically symmetric families of solutions to five-dimensional relativity have been investigated by other authors [6],[7],[8].

3 The $(4 + d)$ -dimensional picture

In the isotropic coordinates we used, the standard radial coordinate $R(r)$ and the radius of compactification $R_i(r)$ of the generic i -th extra dimension are, respectively, given by

$$R(r) = r \left(1 + \frac{\eta}{2r}\right)^2 \left| \frac{1 - \frac{\eta}{2r}}{1 + \frac{\eta}{2r}} \right|^{1 - \frac{m + \sum_i \sigma_i}{\eta}} \quad (10)$$

$$R_i(r) = \rho_i \left| \frac{1 - \frac{\eta}{2r}}{1 + \frac{\eta}{2r}} \right|^{\frac{\sigma_i}{\eta}} \quad (11)$$

The requirement that $R(r)$ be a monotonic function of the radial coordinate r fixes a minimum allowed value r_{\min} for r given by

$$r_{\min} = \begin{cases} \frac{\eta}{2} & \text{for } m + \sum_i \sigma_i < \eta \\ \frac{1}{2} \left[m + \sum_i \sigma_i + \sqrt{\left(m + \sum_i \sigma_i\right)^2 - \eta^2} \right] & \text{for } m + \sum_i \sigma_i \geq \eta \end{cases} \quad (12)$$

One can easily check that the value r_{\min} , say r_0 , which corresponds to the second choice in Eq.(12) never becomes smaller than the value $\frac{\eta}{2}$ and equals it when $m + \sum_i \sigma_i = \eta$.

Let us consider the following cases:

a) $m + \sum_i \sigma_i < \eta$

The solutions display, if m is different from zero, infinite red-shift at $r = \frac{\eta}{2}$. Due however to the fact that $R\left(\frac{\eta}{2}\right) = 0$, the corresponding surface area vanishes so we are in front of naked singularities and cannot even speak of black-holes. On the contrary, each radius $R_i(r)$ at $r = \frac{\eta}{2}$ either vanishes or equals ρ_i or blows up to infinity depending on the value of the ratio $\frac{\sigma_i}{\eta}$. This opens the possibility for extra dimensions, which do not appear because they

are compactified and unobservable at the available energies, not only to be visible near the naked singularity, but else to become there the only sensible spatial dimensions.

b) $m + \sum_i \sigma_i > \eta$

The line element in Eq.(8) can be written in the standard form as

$$ds_{4+d}^2 = - \exp[\phi(R)] dt^2 + \frac{dR^2}{1 - \frac{b(R)}{R}} + R^2 d\Omega^2 + \sum_i \frac{R_i^2(r(R))}{\rho_i^2} dy_i dy_i \quad (13)$$

where

$$\phi(R) = \frac{m}{\eta} \ln \left| \frac{1 - \frac{\eta}{2r(R)}}{1 + \frac{\eta}{2r(R)}} \right| \quad (14)$$

$$1 - \frac{b(R)}{R} = \left[\frac{r^2(R) - (m + \sum_i \sigma_i) r(R) + \frac{\eta^2}{4}}{r^2(R) - \frac{\eta^2}{4}} \right]^2 \quad (15)$$

and $r(R)$ is the inverse of $R(r)$.

The function $\phi(R)$ is finite everywhere, while the function $b(R)$ satisfies the conditions $\frac{b(R)}{R} \leq 1$ and $\frac{b(R)}{R} \rightarrow 0$ as $R \rightarrow \infty$. Moreover when $R(r)$ reaches its minimum value $R_0 = R(r_0)$ it vanishes the right-hand side of Eq.(15) and consequently $\frac{b(R)}{R} \rightarrow 1$ as $R \rightarrow R_0$.

It then follows that $\phi(R)$ and $b(R)$ play, respectively, the role of the redshift function and of the shape function of a wormhole, and R_0 represents the wormhole throat [9],[10],[11]. The radii of the extra dimensions remain finite and different from zero.

c) $m + \sum_i \sigma_i = \eta$

If m is different from zero, the solutions display again infinite red-shift at $r = \frac{\eta}{2}$, but now $R\left(\frac{\eta}{2}\right) = 2(m + \sum_i \sigma_i)$, so the corresponding surface area does not vanish and represents an event horizon. We emphasize here the possibility to have black-holes endowed with scalar charges. If each scalar charge

were equal to zero, then the black-holes would be similar to the Schwarzschild ones.

If m is equal to zero, the solutions represent wormholes as described above. Irrespective of m , the behavior of each radius $R_i(r)$ at $r = \frac{\eta}{2}$ depends only on the value of the ratio $\frac{\sigma_i}{\eta}$.

4 The four-dimensional picture

In order to exhibit the physical meaning of the parameters used till now, we conformally rescale (see for instance [12]) the $(4 + d)$ -dimensional metric

$$ds_{4+d}^2 = \left| \frac{1 - \frac{\eta}{2r}}{1 + \frac{\eta}{2r}} \right|^{-\frac{\sum_i \sigma_i}{\eta}} \left\{ ds_4^2 + \sum_i \left| \frac{1 - \frac{\eta}{2r}}{1 + \frac{\eta}{2r}} \right|^{\frac{2\sigma_i + \sum_j \sigma_j}{\eta}} dy^i dy^i \right\} \quad (16)$$

where ds_4^2 is the four-dimensional space-time metric:

$$\begin{aligned} ds_4^2 = & - \left| \frac{1 - \frac{\eta}{2r}}{1 + \frac{\eta}{2r}} \right|^{\frac{2m + \sum_i \sigma_i}{\eta}} dt^2 \\ & + \left(1 + \frac{\eta}{2r} \right)^4 \left| \frac{1 - \frac{\eta}{2r}}{1 + \frac{\eta}{2r}} \right|^{2 - \frac{2m + \sum_i \sigma_i}{\eta}} (dr^2 + r^2 d\Omega^2) \end{aligned} \quad (17)$$

It is apparent that the physical mass M is now given by

$$M = m + \frac{1}{2} \sum_i \sigma_i \quad (18)$$

which must be a nonnegative quantity.

Every time an extra dimension i is reduced, a massless scalar field φ_i is

defined by the identity

$$\exp \left\{ 2 \left| \frac{2\sigma_i + \sum_j \sigma_j}{\sigma_i} \right|^{\frac{1}{2}} \varphi_i \right\} = \left| \frac{1 - \frac{\eta}{2r}}{1 + \frac{\eta}{2r}} \right|^{\frac{2\sigma_i + \sum_j \sigma_j}{\eta}} \quad (19)$$

whence

$$\varphi_i(r) = \varepsilon_i \frac{|\sigma_i(2\sigma_i + \sum_j \sigma_j)|^{\frac{1}{2}}}{2\eta} \ln \left| \frac{1 - \frac{\eta}{2r}}{1 + \frac{\eta}{2r}} \right| \quad (20)$$

Here $\varepsilon_i = \text{sign}(2\sigma_i + \sum_j \sigma_j)$.

From the four-dimensional Einstein's equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ it is straightforward to calculate the energy-momentum tensor $T_{\mu\nu}$ which can be written as

$$T_{\mu\nu} = \frac{1}{4\pi} \sum_i \left(\nabla_\mu \varphi_i \nabla_\nu \varphi_i - \frac{1}{2} g_{\mu\nu} \nabla^\lambda \varphi_i \nabla_\lambda \varphi_i \right) \quad (21)$$

If we recall Eq.(20), the energy-momentum tensor acquires the form

$$T_{\mu\nu} = \frac{1}{4\pi} \left(\nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} \nabla^\lambda \Phi \nabla_\lambda \Phi \right) \quad (22)$$

having defined a single effective scalar field $\Phi(r)$ as

$$\Phi(r) = \frac{\sigma}{\eta} \ln \left| \frac{1 - \frac{\eta}{2r}}{1 + \frac{\eta}{2r}} \right| \quad (23)$$

and a related scalar charge σ given by

$$\sigma^2 = \sum_i \frac{\sigma_i(2\sigma_i + \sum_j \sigma_j)}{4} = \frac{2 \sum_i \sigma_i^2 + (\sum_i \sigma_i)^2}{4} \quad (24)$$

The parameter η defined by Eq.(7) reduces simply to

$$\eta = \sqrt{M^2 + \sigma^2} \quad (25)$$

and the line element in Eq.(17) becomes

$$ds_4^2 = - \left| \frac{1 - \frac{\eta}{2r}}{1 + \frac{\eta}{2r}} \right|^{2\frac{M}{\eta}} dt^2 + \left(1 + \frac{\eta}{2r}\right)^4 \left| \frac{1 - \frac{\eta}{2r}}{1 + \frac{\eta}{2r}} \right|^2 \left(1 - \frac{M}{\eta}\right) (dr^2 + r^2 d\Omega^2) \quad (26)$$

This line element, where the singularity at $r = \frac{\eta}{2}$ is naked, was obtained in past years starting from four-dimensional general relativity plus a massless scalar field (see, for instance, [13] and references quoted therein).

5 Conclusions

In this brief paper we have treated some features of a $(4 + d)$ dimensional Kaluza-Klein theory governed by the vacuum field equations.

We found that the higher-dimensional theory can describe not only black-holes and naked singularities but, what seems a remarkable propriety, also wormholes. Reducing by a conformal rescaling to the conventional four-dimensional theory in which gravity is coupled to a massless scalar field, only naked singularities can survive because here holds the no-hair theorem and are not violated the energy conditions which prevent the formation of wormholes [14].

As a concluding remark, we notice that our solutions would remain the same if changing the signature of the extra dimensions from space-like to time-like. It is well known that time-like extra dimensions give rise to problems of causality or of insufficient predictability [15], nevertheless some of these drawbacks might be overcome by the fact that extra dimensions are compactified [16].

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